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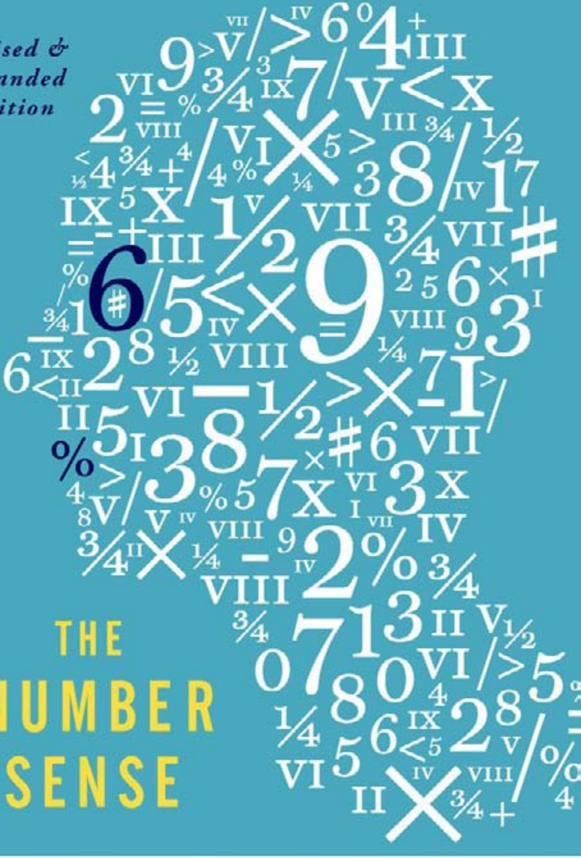

THE NUMBER SENSE

[HOW THE MIND CREATES MATHEMATICS]

STANISLAS DEHEANE

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The Number Sense

HOW THE MIND CREATES MATHEMATICS

Revised and Updated Edition

Stanislas Dehaene

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UNIVERSITY PRESS

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To Ghislaine, Oliver, David, and Guillaume

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Preface to the Second Edition

A SCIENTIFIC BOOK is an unintentional time capsule. It has no sell-by date, which often means that readers will evaluate its theories, facts, and evidence, many years after publication, and do so with the omniscience of hindsight. *The Number Sense*, a book I wrote fifteen years ago, in my late twenties, is no exception to this rule.

I was lucky to start work on *The Number Sense* in the early 1990s, at a time when number research was in its infancy. A handful of laboratories had only just begun to scratch the surface of the field. Some focused on how infants perceived sets of objects. Others specialized in the way schoolchildren learn their multiplication tables, or studied the bizarre behavior of patients suffering from brain lesions that disrupted calculation. Finally, some, like me, made the first forays into brain imaging research to find out which brain areas lit up when students were asked a simple arithmetic question, like, is 6 larger than 5? Only a few of us, at the time, could see how all these studies would one day be pulled together into a single field, mathematical cognition, with multifaceted techniques all aimed at answering Warren McCulloch's stimulating query:

> *"What is a number, that a man may know it, and a man, that he may know a number?"*

The Number Sense was written with this single goal in mind: to assemble all the available facts on how the brain does elementary arithmetic, and prove that a new and promising field of research, ripe with empirical findings, was dawning. I also hoped that it might, perhaps, shed light on ancient philosophical disputes that questioned the very nature of mathematics. During the three years that it took me to put together all the different lines of research in the field, my enthusiasm increased as I realized how all the pieces of this complex puzzle fitted together into a coherent whole. Animal research on number pointed to an age-old competence for processing approximate quantities. This "number sense," which is also present in infants, gave humans the intuition of number. Cultural inventions, such as the abacus or Arabic numerals, then transformed it into our fully-fledged capacity for symbolic mathematics. It was therefore obvious that a careful look at the brain structures for the number sense could shed much light on our understanding of mathematics. It provided a clear view of how evolution had proceeded, and reconnected our human abilities for mathematics to the way monkeys' and even rats' and pigeons' brains represent numbers.

Since this book was written, some fifteen years ago, a flurry of innovative research has given this area a stronger impetus that I ever imagined. Mathematical cognition is now a well-established domain in cognitive science, and is no longer centered exclusively on the concept of number and its origins but has expanded into the related domains of algebra and geometry. Several research topics that were merely outlined in *The Number Sense* have become fully-fledged areas of research: number sense in animals, brain imaging of numerical computations, the nature of the impairment in children with mathematical difficulties… One of the most exciting breakthroughs has been the discovery of single neurons that code for number in the monkey brain, at a precise site in the parietal lobe that appears to be a plausible homolog of the human area that activates when we calculate. Another rapidly developing area has to do with the application of this knowledge to education: we are beginning to understand how schooling develops the understanding of exact number and arithmetic, and how children who are at risk of developing dyscalculia can be helped with very simple games and software.

When I reread the first edition of this book, I was pleased to see that all of these ideas were already germinating, albeit somewhat speculatively, fifteen years ago. Now that research findings have solidly grounded them, I am convinced that a new edition of *The Number Sense* is in order. To be sure, several excellent books had been published since 1997, among them Brian Butterworth's *Mathematical Brain* (1999), Rafael Núñez and George Lakoff's *Where Mathematics Comes From* (2000), and Jamie Campbell's edited *Handbook of Mathematical Cognition* (2004). But none of them captures the full range of what we understand today about number and the brain.

I am grateful to my agents, Max and John Brockman, and to my editors, Abby Gross and Odile Jacob, for encouraging me to embark on this new version and for helping me to decide what form it should take. We quickly agreed that to rewrite the past would be awkward or even presumptuous. It seemed important to give the reader an appropriate sense of how the field came into being twenty years ago, what motivated our current hypotheses, and how experimental methods had evolved since then, either to flesh out our theories—or, occasionally, but fortunately not too often, to refute them. Thus, we conceived a second edition that would leave the original untouched but would supplement it with new references and, above all, a long, new, final chapter outlining the most outstanding discoveries that have been made since the first edition appeared. Selecting the findings that belonged in this chapter was an arduous task, since the field has literally exploded in the last fifteen years. Indeed, there are now hundreds of scientific findings that would have been relevant. Nevertheless, I decided to stick to a small list of surprising facts that, I believe, illuminate what arithmetic is at the brain level, and therefore how we should teach it.

Most mathematicians, overtly or covertly, are Platonists. They picture themselves as explorers of a continent of ideas independent of the human mind, older than life and immanent in the very structure of the Universe. In his treatise on *The Nature and Meaning of Numbers*, the great mathematician Richard Dedekind, however, thought otherwise. Numbers, he said, are "free creations of the human mind," "an immediate emanation from the pure laws of thought." I could not agree more—but then the burden of elucidation clearly falls upon psychologists and neuroscientists, who will have to figure out how a finite brain, a mere collection of nerve cells, can conceive such abstract thoughts. The present book should be considered as a modest contribution to this fascinating question.

S.D.
Palaiseau, France
July 2010

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Preface to the First Edition

WE ARE SURROUNDED by numbers. Etched on credit cards or engraved on coins, printed on pay checks or aligned on computerized spread sheets, numbers rule our lives. Indeed, they lie at the heart of our technology. Without numbers, we could not send rockets roaming the solar system, nor could we build bridges, exchange goods, or pay our bills. In some sense, then, numbers are cultural inventions only comparable in importance to agriculture or to the wheel. But they might have even deeper roots. Thousands of years before Christ, Babylonian scientists used clever numerical notations to compute astronomical tables of amazing accuracy. Tens of thousands of years prior to them, Neolithic men recorded the first written numerals by engraving bones or by painting dots on cave walls. And, as I shall try to convince you later on, millions of years earlier still, long before the dawn of humankind, animals of all species were already registering numbers and entering them into simple mental computations. Might numbers, then, be almost as old as life itself? Might they be engraved in the very architecture of our brains? Do we all possess a "number sense," a special intuition that helps us make sense of numbers and mathematics?

Around the age of sixteen, as I was training to become a mathematician, I became fascinated by the abstract objects I was taught to manipulate, and above all by the simplest of them—numbers. Where did they come from? How was it possible for my brain to understand them? Why did it seem so difficult for most people to master them? Historians of science and philosophers of mathematics had provided some tentative answers, but to a scientifically oriented mind their speculative and contingent character was unsatisfactory Furthermore, scores of intriguing facts about numbers and mathematics were left unanswered in the books I knew of. Why did all languages have at least some number names? Why did everybody seem to find multiplications by seven, eight, or nine particularly hard to learn? Why couldn't I seem to recognize more than four objects at a glance? Why were there ten boys for one girl in the high-level mathematics classes I was attending? What tricks allowed lightning calculators to multiply two three-digit numbers in a few seconds?

As I learned increasingly more about psychology, neurophysiology, and computer science, it became obvious that the answers had to be looked for, not in history books, but in the very structure of our brains—the organ that enables us to create mathematics. It was an exciting time for a mathematician to turn to cognitive neuroscience. New experimental techniques and amazing results seemed to appear every month. Some revealed that animals could do simple arithmetic. Others asked whether babies had any notion of 1 plus 1. Functional imaging tools were also becoming available that could visualize the active circuits of the human brain as it calculates and solves arithmetical problems. Suddenly, the psychological and cerebral bases of our number sense were open to experimentation. A new field of science was emerging: mathematical cognition, or the scientific inquiry into how the human brain gives rise to mathematics. I was lucky enough to become an active participant in this quest. This book provides a first glance at this new field of research that my colleagues in Paris, and several research teams throughout the world, are still busy developing.

I am indebted to many people for helping me complete the transition from mathematics to neuropsychology. First and foremost, my research program on arithmetic and the brain could never have developed without the generous assistance of three outstanding teachers, colleagues, and friends who deserve very special thanks: Jean-Pierre Changeux in neurobiology, Laurent Cohen in neuropsychology, and Jacques Mehler in cognitive psychology. Their support, advice, and often direct contribution to the work described here have been of invaluable help.

I would like to acknowledge my many research companions of the past two decades, and particularly the crucial contribution of the many students and post-docs, many of whom became essential collaborators and, quite simply, friends that count: Rokny Akhavein, Serge Bossini, Marie Bruandet, Antoine Del Cul, Raphaël Gaillard, Pascal Giraux, Ed Hubbard, Véronique Izard, Markus Kiefer, André Knops, Étienne Kœchlin, Sid Kouider, Gurvan Leclec'H, Cathy Lemer, Koleen McCrink, Nicolas Molko, Lionel Naccache, Manuela Piazza, Philippe Pinel, Maria-Grazia Ranzini, Susannah Revkin, Gérard Rozsavolgyi, Elena Rusconi, Mariano Sigman, Olivier Simon, Arnaud Viarouge, and Anna Wilson.

For the first edition of this book, I also benefited from the advice of many other eminent scientists. Mike Posner, Don Tucker, Michael Murias, Denis Le Bihan, André Syrota, and Bernard Mazoyer shared with me their in-depth knowledge of brain imaging. Emmanuel Dupoux, Anne Christophe, and Christophe Pallier advised me in psycholinguistics. I am also grateful for ground-shaking debates with Rochel Gelman and Randy Gallistel, and for judicious remarks by Karen Wynn, Sue Carey, and Josiane Bertoncini on child development. The late professor Jean-Louis Signoret had introduced me to the fascinating domain of neuropsychology. Subsequently, numerous discussions with Alfonso Caramazza, Michael McCloskey, Brian Butterworth, and Xavier Seron greatly enhanced my understanding of this discipline. Xavier Jeannin and Michel Dutat, finally, assisted me in programming my experiments.

For this second edition, many additional collaborators, in France and abroad, helped me progress in my research: Hillary Barth, Eliza Block, Jessica Cantlon, Laurent Cohen Jean-Pierre Changeux, Evelyn Eger, Lisa Feigenson, Guillaume Flandin, Tony Greenwald, Marc Hauser, Antoinette Jobert, Ferath Kherif, Andrea Patalano, Lucie Hertz-Pannier, Karen Kopera-Frye, Denis Le Bihan, Stéphane Lehéricy, Jean-François Mangin, J. Frederico Marques, Jean-Baptiste Poline, Denis Rivière, Jérôme Sackur, Elizabeth Spelke, Ann Streissguth, Bertrand Thirion, Pierre-François van de Moortele, and Marco Zorzi. I also gratefully acknowledge all the colleagues who, across the years and the oceans, through relentless discussions, helped me sharpen my thoughts and correct my errors. An exhaustive list is impossible, but my thoughts go first and foremost to Elizabeth Brannon, Wim Fias, Randy Gallistel, Rochel Gelman, Usha Goswami, Nancy Kanwisher, Andreas Nieder, Michael Posner, Bruce McCandliss, Sally and Bennett Shaywitz, and Herb Terrace.

My research on numerical cognition received a massive boost when I received a ten-year Centennial Fellowship grant from the McDonnell Foundation, which played an essential role in my career. It was also supported by INSERM (French Institute for Health and Medical Research, CEA (Atomic Energy Commission), Collège de France, Paris XI University, the Fyssen foundation, the Bettencourt-Schueller Foundation, the Volkswagen foundation, the Louis D. Foundation of the Institut de France, and the French Foundation for Medical Research. The preparation of this book greatly benefited from the close scrutiny of Brian Butterworth, Robbie Case, Markus Giaquinto, and Susana Franck for the English edition, and of Jean-Pierre Changeux, Laurent Cohen, Ghislaine Dehaene-Lambertz and Gérard Jorland for the French edition. Warm thanks go also to Joan Bossert and Abby Gross, my editors at Oxford University Press, John Brockman, my agent, and Odile Jacob, my French editor. Their trust and support was very precious.

I would also like to thank the publishers and authors who kindly granted me the permission to reproduce the figures and quotes used in this book. Special thanks go to Gianfranco Denes for drawing my attention to the remarkable section of Ionesco's *Lesson* that is cited in Chapter 8.

Last but not least, a word of thanks cannot suffice to express my feelings for my family, Ghislaine, Oliver, David, and Guillaume, who patiently supported me during the long months spent exploring and writing about the universe of numbers. This book is dedicated to them.

S.D.
Piriac, France
August 1996

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Any poet, even the most allergic to
mathematics, has to count up to twelve
in order to compose an alexandrine.

RAYMOND QUENEAU

Introduction

AS I FIRST sat down to write this book, I was faced with a ridiculous problem of arithmetic: If this book is to have 250 pages and nine main chapters, how many pages will each chapter have? After thinking hard, I came to the conclusion that each should have slightly fewer than 30 pages. This took me about five seconds, not bad for a human, yet an eternity compared to the speed of any electronic calculator. Not only did my calculator respond instantaneously, but the result it gave was accurate to the tenth decimal: 27.7777777778!

Why is our capacity for mental calculation so inferior to that of computers? And how do we reach excellent approximations such as "slightly fewer than 30" without resorting to an exact calculation, something that is beyond the best of electronic calculators? The resolution of these nagging questions, which is the subject matter of this book, will confront us with even more challenging riddles:

- Why is it that after so many years of training, the majority of us still do not know for sure whether 7 times 8 is 54 or 64... or is it 56?
- Why is our mathematical knowledge so vulnerable that a small cerebral lesion is enough to abolish our sense of numbers?
- How can a 5-month-old baby know that 1 plus 1 equals 2?
- How is it possible for animals without language, such as chimpanzees, rats, and pigeons, to have some knowledge of elementary arithmetic?

My hypothesis is that the answers to all these questions must be sought at a single source: the structure of our brain. Every single thought we entertain, every calculation we perform, results from the activation of specialized neuronal circuits implanted in our cerebral cortex. Our abstract mathematical constructions originate in the coherent activity of our cerebral circuits, and of the millions of other brains preceding us that helped shape and select our current mathematical tools. Can we begin to understand the constraints that our neural architecture imposes on our mathematical activities?

Evolution, ever since Darwin, has remained the reference for biologists. In the case of mathematics, both biological and cultural evolution matter. Mathematics is not a static and God-given ideal, but an ever-changing field of human research. Even our digital notation of numbers, as obvious as it may seem now, is the fruit of a slow process of invention over thousands of years. The same holds for the current multiplication algorithm, the concept of square root, the sets of real, imaginary, or complex numbers, and so on. All still bear scars of their difficult and recent birth.

The slow cultural evolution of mathematical objects is a product of a very special biological organ, the brain, that itself represents the outcome of an even slower biological evolution governed by the principles of natural selection. The same selective pressures that have shaped the delicate mechanisms of the eye, the profile of the hummingbird's wing, or the minuscule robotics of the ant, have also shaped the human brain. From year to year, species after species, ever more specialized mental organs have blossomed within the brain to better process the enormous flux of sensory information received, and to adapt the organism's reactions to a competitive or even hostile environment.

One of the brain's specialized mental organs is a primitive number processor that prefigures, without quite matching it, the arithmetic that is taught in our schools. Improbable as it may seem, numerous animal species that we consider stupid or vicious, such as rats and pigeons, are actually quite gifted at calculation. They can represent quantities mentally and transform them according to some of the rules of arithmetic. The scientists who have studied these abilities believe that animals possess a mental module, traditionally called the "accumulator," that can hold a register of various quantities. We shall see later how rats exploit this mental accumulator to distinguish series of two, three, or four sounds, or to compute approximate additions of two quantities. The accumulator mechanism opens up a new dimension of sensory perception through which the cardinal of a set of objects can be perceived just as easily as their color, shape, or position. This "number sense" provides animals and humans alike with a direct intuition of what numbers mean.

Tobias Dantzig, in his book exalting "number, the language of science," underlined the primacy of this elementary form of numerical intuition: "Man, even in the lower stages of development, possesses a faculty which, for want of a better name, I shall call *Number Sense.* This faculty permits him to recognize that something has changed in a small collection when, without his direct knowledge, an object has been removed or added to the collection."[1]

[1] Dantzig, 1967.

Dantzig wrote these words in 1954, when psychology was dominated by Jean Piaget's theory, which denied young children any numerical abilities. It took twenty more years before Piagetian constructivism was definitely refuted and Dantzig's insight was confirmed. All people possess, even within their first year of life, a well-developed intuition about numbers. Later, we consider in some detail the ingenious experiments which demonstrate that human babies, far from being helpless, already know right from birth some fragments of arithmetic comparable to the animal knowledge of number. Elementary additions and subtractions are already available to 6-month-old babies!

Let there be no misunderstanding. Obviously, only the adult *Homo sapiens* brain has the power to recognize that 37 is a prime number, or to calculate approximations of the number π. Indeed, such feats remain the privilege of only a few humans in a few cultures. The baby brain and *a fortiori* the animal brain, far from exhibiting our mathematical flexibility, work their minor arithmetical miracles only within quite limited contexts. In particular, their accumulator cannot handle discrete quantities, but only continuous estimates. Pigeons will never be able to distinguish 49 from 50, because they cannot represent these quantities other than in an approximate and variable fashion. For an animal, 5 plus 5 does not make 10, but only *about 10:* maybe 9, 10, or 11. Such poor numerical acuity, such fuzziness in the internal vision of numbers, prevents the emergence of exact arithmetical knowledge in animals. By the very structure of their brains, they are condemned to an approximate arithmetic.

Humans, however, have been endowed by evolution with a supplementary competence: the ability to create complex symbol systems, including spoken and written language. Words or symbols, because they can separate concepts with arbitrarily close meanings, allow us to move beyond the limits of approximation. Language allows us to label infinitely many different numbers. These labels, the most evolved of which are the Arabic numerals, can symbolize and discretize any continuous quantity. Thanks to them, numbers that may be close in quantity, but whose arithmetical properties are very different, can be distinguished. Only then can the invention of purely formal rules for comparing, adding, or dividing two numbers be conceived. Indeed, numbers acquire a life of their own, devoid of any direct reference to concrete sets of objects. The scaffolding of mathematics can then rise, ever higher, ever more abstract.

This raises a paradox, however. Our brains have remained essentially unchanged since *Homo sapiens* first appeared 100,000 years ago. Our genes, indeed, are condemned to a slow and minute evolution, dependent on the occurrence of chance mutations. It takes thousands of aborted attempts before a favorable mutation, one worthy of being passed on to coming generations, emerges from the noise. In contrast, cultures evolve through a much faster process. Ideas, inventions, progress of all kinds, can spread to an entire population through language and education as soon as they have germinated in some fertile mind. This is how mathematics, as we know it today, has emerged in only a few thousand years. The concept of number, hinted at by the Babylonians, refined by the Greeks, purified by the Indians and the Arabs, axiomatized by Dedekind and Peano, generalized by

Galois, has never ceased to evolve from culture to culture—obviously, without requiring any modification of the mathematician's genetic material! In a first approximation, Einstein's brain is no different from that of the master who, in the Magdalenian, painted the Lascaux cave. At elementary school, our children learn modern mathematics with a brain initially designed for survival in the African savanna.

How can we reconcile such biological inertia with the lightning speed of cultural evolution? Thanks to extraordinary modern tools, such as positron emission tomography or functional magnetic resonance imaging, the cerebral circuits that underlie language, problem solving, and mental calculation can now be imaged in the living human brain. We will see that when our brain is confronted with a task for which it was not prepared by evolution, such as multiplying two digits, it recruits a vast network of cerebral areas whose initial functions are quite different, but which may, together, reach the desired goal. Aside from the approximate accumulator that we share with rats and pigeons, our brain probably does not contain any "arithmetical unit" predestined for numbers and math. It compensates this shortcoming, however, by tinkering with alternative circuits that may be slow and indirect, but are more or less functional for the task at hand.

Cultural objects—for instance, written words or numbers—may thus be considered as parasites that invade cerebral systems initially destined to a quite different use. Occasionally, as in the case of word reading, the parasite can be so intrusive as to completely replace the previous function of a given brain area with its own. Thus, some brain areas that, in other primates, seem to be dedicated to the recognition of visual objects acquire in the literate human a specialized and irreplaceable role in the identification of letter and digit strings.

One cannot but marvel at the flexibility of a brain that can, depending on context and epoch, plan a mammoth hunt or conceive of a demonstration of Fermat's last theorem. However, this flexibility should not be overestimated. Indeed, my contention is that it is precisely the assets and the limits of our cerebral circuits that determine the strong and weak points of our mathematical abilities. Our brain, like that of the rat, has been endowed since time immemorial with an intuitive representation of quantities. This is why we are so gifted for approximation, and why it seems so obvious to us that 10 is larger than 5. Conversely, our memory, unlike that of the computer, is not digital but works by association of ideas. This is probably the reason why we have such a hard time remembering the small number of equations that make up the multiplication table.

Just as the budding mathematician's brain thus lends itself more or less easily to the requirements of mathematics, mathematical objects also evolve to match our cerebral constraints increasingly well. The history of mathematics provides ample evidence that our concepts of number, far from being frozen, are in constant evolution. Mathematicians have worked hard for centuries to improve the usefulness of numerical notations by increasing their generality, their fields of application, and their formal simplicity. In doing so, they have unwittingly invented ways of making them fit the constraints of our cerebral organization. Though a few years of education now suffice for a child to learn digital notation, we should not forget that it took centuries to perfect this system before it became child's play. Some mathematical objects now seem very intuitive only because their structure is well adapted to our brain architecture. On the other hand, a great many children find fractions very difficult to learn because their cortical machinery resists such a counterintuitive concept.

If the basic architecture of our brain imposes such strong limits on our understanding of arithmetic, why do a few children thrive on mathematics? How have outstanding mathematicians such as Gauss, Einstein, or Ramanujan attained such extraordinary familiarity with mathematical objects? And how do some idiot savants with an IQ of 50 manage to become experts in mental calculation? Do we have to suppose that some people started in life with a particular brain architecture, or a biological predisposition to become geniuses? A careful examination of this supposition will show us that this is unlikely. At present, at any rate, very little evidence exists that great mathematicians and calculating prodigies have been endowed with an exceptional neurobiological structure. Like the rest of us, experts in arithmetic have to struggle with long calculations and abstruse mathematical concepts. If they succeed, it is only because they devote a considerable time to this topic and eventually invent well-tuned algorithms and clever shortcuts that any of us could learn if we tried, and that are carefully devised to take advantage of our brain's assets and get round its limits. What is special about them is their disproportionate and relentless passion for numbers and mathematics, occasionally fueled by their inability to entertain normal relations with other fellow humans, a cerebral disease called *autism*. I am convinced that children of equal initial abilities may become excellent or hopeless at mathematics depending on their love or hatred of the subject. Passion breeds talent—and parents and teachers, therefore, have a considerable responsibility in developing their children's positive or negative attitudes toward mathematics.

In *Gulliver's Travels,* Jonathan Swift describes the bizarre teaching methods used at the mathematics school of Lagado, in Balnibarbi Island:

> I was at the mathematical school, where the master taught his pupils after a method scarcely imaginable to us in Europe. The proposition and demonstration were fairly written on a thin wafer, with ink composed of a cephalic tincture. This the student was to swallow upon a fasting stomach, and for three days following eat nothing but bread and water. As the wafer digested, the tincture mounted to his brain, bearing the proposition along with it. But the success hath not hitherto been answerable, partly by some error in the *quantum* or composition, and partly by the perverseness of lads, to whom this bolus is so nauseous, that they generally steal aside, and discharge it upwards before it can operate; neither have they been yet persuaded to use so long an abstinence as the prescription requires.

Although Swift's description reaches the height of absurdity, his basic metaphor of learning mathematics as a process of assimilation has an undeniable truth. In the final analysis, all mathematical knowledge is incorporated into the biological tissues of the brain. Every single mathematics course that our children take is made possible by the modifications of millions of their synapses, implying widespread gene expression and the formation of billions of molecules of neurotransmitters and receptors, with modulation by chemical signals reflecting the child's level of attention and emotional involvement in the topic. Yet the neuronal networks of our brains are not perfectly flexible. The very structure of our brain makes certain arithmetical concepts easier to "digest" than others.

I hope that the views I am defending here will eventually lead to improvements in teaching mathematics. A good curriculum would take into account the assets and limits of the learner's cerebral structure. To optimize the learning experiences of our children, we should consider what impact education and brain maturation have on the organization of mental representations. Obviously, we are still far from understanding to what extent learning can modify our brain machinery. The little that we already know could be of some use, however. The fascinating results that cognitive scientists have accumulated for the last twenty years on how our brain does math have not, until now, been made public and allowed to percolate through to the world of education. I would be delighted if this book served as a catalyst for improved communication between the cognitive and education sciences.

This book will take you on a tour of arithmetic as seen from the eyes of a biologist, but without neglecting its cultural components. In Chapters 1 and 2, through an initial visit of animals' and human infants' abilities for arithmetic, I shall try to convince you that our mathematical abilities are not without biological precursors. Indeed, in Chapter 3 we shall find many traces of the animal mode of processing numbers still at work in adult human behavior. In Chapters 4 and 5, by observing how children learn to count and to calculate, we shall then attempt to understand how this initial approximate system can be overcome, and the difficulties that the acquisition of advanced mathematics raises for our primate brain. This will be a good occasion to investigate current methods of mathematical teaching and to examine the extent to which they have naturally adapted to our mental architecture. In Chapter 6 we shall also try to sort out the characteristics that distinguish a young Einstein or a calculating prodigy from the rest of us. In Chapters 7 and 8, finally, our number hunt will end up in the fissures of the cerebral cortex, where the neuronal circuits that support calculation are located, and from which, alas, they can be dislodged by a lesion or a vascular accident, thus depriving otherwise normal persons of their number sense.

1

Our Numerical Heritage

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One stone
two houses
three ruins
four gravediggers
one garden
some flowers

one raccoon

JACQUES PRÉVERT, *Inventaire*

1

TALENTED AND GIFTED ANIMALS

BOOKS ON NATURAL history have recounted the following anecdote since the eighteenth century:

> A nobleman wanted to shoot down a crow that had built its nest atop a tower on his domain. However, whenever he approached the tower, the bird flew out of gun range and waited until the man departed. As soon as he left, it returned to its nest. The man decided to ask a neighbor for help. The two hunters entered the tower together, and later only one of them came out. But the crow did not fall into this trap, and carefully waited for the second man to come out before returning. Neither did three, then four, then five men fool the clever bird. Each time, the crow would wait until all the hunters had departed. Eventually, the hunters came as a party of six. When five of them had left the tower, the bird, not so numerate after all, confidently came back, and was shot down by the sixth hunter.

Is this anecdote authentic? Nobody knows. It is not even clear that it has anything to do with numerical competence: For all we know, the bird could have memorized the visual appearance of each hunter rather than their number. Nevertheless, I decided to highlight it because it provides a splendid illustration of many aspects of animal arithmetic that are the subject of this chapter. First, in many tightly controlled experiments, birds and many other animal species appear to be able to perceive numerical quantities without requiring special training. Second, this perception is not perfectly accurate, and its accuracy decreases with increasingly larger numbers; hence the bird confounding 5 and 6. Finally, and more facetiously, the anecdote shows how the forces of Darwinian selection also apply to the arithmetical domain. If the bird had been able to count up to 6, perhaps it would never have been shot! In numerous species, estimating the number and ferocity of predators, or quantifying and comparing the return of two sources of food, are matters of life and death. Such evolutionary arguments should help make sense of the many scientific experiments that have revealed sophisticated procedures for numerical calculation in animals.

A Horse Named Hans

At the beginning of this century, a horse named Hans made it to the headlines of German newspapers.[1] His master, Wilhelm von Osten, was no ordinary circus animal trainer. Rather, he was a passionate man who, under the influence of Darwin's ideas, had set out to demonstrate the extent of animal intelligence. He wound up spending more than a decade teaching his horse arithmetic, reading, and music. Although the results were slow to come, they eventually exceeded all his expectations. The horse seemed gifted with a superior intelligence. It could apparently solve arithmetical problems and even spell out words!

Demonstrations of Clever Hans's abilities often took place in von Osten's yard. The public would form a half-circle around the animal and suggest an arithmetical question to the trainer—for instance, "How much is 5 plus 3?" Von Osten would then present the animal with five objects aligned on a table, and with three other objects on another table. After examining the "problem," the horse responded by knocking on the ground with its hoof the number of times equal to the total of the addition. However, Hans's mathematical abilities far exceeded this simple feat. Some arithmetical problems were spoken aloud by the public, or were written in digital notation on a blackboard, and Hans could solve them just as easily (Figure 1.1). The horse could also add two fractions such as 2/5 and 1/2 and give the answer 9/10 by striking nine times, then ten times with its hoof. It was even said that to the question of determining the divisors of 28, Hans came out very appropriately with the answers 2, 4, 7, 14, and 28. Obviously, Hans's number knowledge surpassed by far what an elementary school teacher would expect today of a reasonably bright pupil!

In September 1904, a committee of experts, among whom figured the eminent German psychologist Carl Stumpf, concluded after an extensive investigation that Hans's feats were real and not a result of cheating. This generous conclusion, however, did not satisfy Oskar Pfungst, one of Stumpf's own students. With von Osten's help—the master was

[1] Fernald, 1984

FIGURE 1.1. Clever Hans and his master Wilhelm von Osten strike a pose in front of an impressive array of arithmetic problems. The larger blackboard shows the numerical coding the horse used to spell words.
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fully convinced of his prodigy's superior intelligence—he began a systematic study of the horse's abilities. Pfungst's experiments, even by today's standards, remain a model of rigor and inventiveness. His working hypothesis was that the horse could not but be totally inept in mathematics. Therefore, it had to be the master himself, or someone in the public, who knew the answer and sent the animal a hidden signal when the target number of strokes had been reached, thus commanding the animal to stop knocking with its hoof.

To prove this, Pfungst invented a way of dissociating Hans's knowledge of a problem from what its master knew. He used a procedure that differed only slightly from the one described above. The master watched carefully as a simple addition was written in large printed characters on a panel. The panel was then oriented toward the horse in such a way that only it could see the problem and answer it. However, on some trials, Pfungst surreptitiously modified the addition before showing it to the horse. For instance, the master could see 6 + 2, whereas in fact the horse was trying to solve 6 + 3.

The results of this experiment, and of a series of follow-up controls, were clear-cut. Whenever the master knew the correct response, Hans got the right answer. When, on the contrary, the master was not aware of the solution, the horse failed. Moreover, the horse often produced an error that matched the numerical result expected by its master. Obviously, it was von Osten himself, rather than Hans, who was finding the solution to the various arithmetical problems. But how then did the horse know how to respond? Pfungst eventually deduced that Hans's truly amazing ability lay in detecting minuscule movements of its master's head or eyebrows that invariably announced the time to stop the series of knocks. In fact, Pfungst never doubted that the trainer was sincere.

He believed that the signals were completely unconscious and involuntary. Even when von Osten was absent, the horse continued to respond correctly: Apparently, it detected the buildup of tension in the public as the expected number of hoof strokes was attained. Pfungst himself could never eliminate all forms of involuntary communication with the animal, even after he discovered the exact nature of the body clues it used.

Pfungst's experiments largely discredited demonstrations of "animal intelligence" and the competence of self-proclaimed experts such as Stumpf who had blindly subscribed to them. Indeed, the "Clever Hans phenomenon" is still taught in psychology classes today. It remains a symbol of the pernicious influence that experimenter expectations and interventions, however small, may have on the outcome of any psychological experiment with humans or with animals. Historically, Hans's story has played a crucial role in shaping the critical minds of psychologists and ethologists. It has drawn attention to the necessity for a rigorous experimental design. Since an essentially invisible stimulation, as brief as the blink of an eye, can influence the performance of animals, a well-designed experiment has to be devoid from the start of any possible source of errors. This lesson was particularly well received by behaviorists, such as B. F. Skinner, who dedicated a large amount of work to the development of rigorous experimental paradigms for the study of animal behavior.

Unfortunately, Hans's exemplary case has also had more negative consequences on the development of psychological science. It has imposed an aura of suspicion onto the whole area of research on the representation of numbers in animals. Ironically, scientists now meet every single demonstration of numerical competence in animals with the same raised eyebrows that served as a cue to Hans! Such experiments are immediately associated, consciously or not, with Hans's story, and are therefore suspected of a basic flaw in design, if not downright forgery. This is an irrational prejudice, however. Pfungst's experiments showed only that Hans's numerical abilities were a fluke. By no means did they prove that it is impossible for an animal to understand some aspects of arithmetic. For a long time, however, the scientist's attitude was to systematically look for some experimental bias that might explain animal behavior without resorting to the hypothesis that animals have even an embryonic knowledge of calculation. For a while, even the most convincing results failed to convince anyone. Some researchers even preferred to attribute to animals mysterious abilities such as a "rhythm discrimination" faculty, for instance, rather than admit that animals could enumerate a collection of objects. In brief, the scientific community tended to throw out the baby with the bath water.

Before turning to some of the experiments that finally convinced all but the most skeptical of researchers, I would like to conclude Hans's story with a modern anecdote. Even today, the training of circus animals rests on methods rather similar to Hans's trick. If you ever see a show in which an animal adds numbers, spells words, or some surprising deed of this kind, you may safely bet that its behavior rests, like Hans's, on a hidden communication with its human trainer. Let me stress again that such communication need not be intentional. The trainer is often sincerely convinced of his pupil's gifts. A few years ago, I came upon an amusing article in a local Swiss newspaper. A journalist had visited the home of Gilles and Caroline P., whose poodle, named Poupette, seemed extraordinarily gifted in mathematics. Figure 1.2 shows Poupette's proud owner presenting his faithful and brilliant companion with a series of written digits that it was supposed to add. Poupette responded without ever making an error by tapping on its master's hand with its paw the exact number of times required, and then licking the hand after the correct count had been reached. According to its master, the canine prodigy had required only a brief training period, which led him to believe in reincarnation or some similar paranormal phenomenon. The journalist, however, wisely noted that the dog could react to subtle cues from the master's eyelids, or to some tiny motions of his hand when the correct count was reached. So this was indeed a case of reincarnation after all: the reincarnation of Clever Hans's stratagem, of which Poupette's story constituted, a century later, an astonishing replication.

Rat Accountants

Following the Hans episode, several renowned American laboratories developed research programs on animal mathematical abilities. Many such projects failed. A famous German ethologist named Otto Koehler, however, was more successful.[2] One of his trained crows, Jacob, apparently learned to choose, among several containers, the one whose lid bore a fixed number of five points. Because the size, the shape, and the location of the points varied randomly from trial to trial, only an accurate perception of the number 5 could

FIGURE 1.2. A modern canine "clever Hans": Poupette, the dog that could supposedly add digits.

[2] Koehler, 1951

account for this performance. Nevertheless, the results achieved by Koehler's team had little impact, partly because most of their results were published only in German, and partly because Koehler failed to convince his colleagues that all possible sources of error, such as unintentional experimenter communication, olfactory cues or the like, had been excluded.

In the 1950s and 1960s, Francis Mechner, an animal psychologist at Columbia University, followed by John Platt and David Johnson at the University of Iowa, introduced a very convincing experimental paradigm that I shall schematically describe here.[3] A rat that had been temporarily deprived of food was placed in a closed box with two levers, A and B. Lever B was connected to a mechanical device that delivered a small amount of food. However, this reward system did not work at once. The rat first had to repeatedly press lever A. Only after it had pressed for a fixed number of times n on lever A could it switch to lever B and get its deserved treat. If the rat switched too early to lever B, not only did it fail to get any food, but it received a penalty. On different experiments, the light could go off for a few seconds, or the counter was reset so that the rat had to start all over again with a new series of n presses on lever A.

How did rats behave in this rather unusual environment? They initially discovered, by trial and error, that food would appear when they pressed several times on lever A, and then once on lever B. Progressively, the number of times that they had to press was estimated more and more accurately Eventually, at the end of the learning period, the rats behaved very rationally in relation to the number n that had been selected by the experimenter. The rats that had to press four times on lever A, before lever B would deliver food, did press it about four times. Those that were placed in the situation where eight presses were required waited until they had produced about eight squeezes, and so on (see Figure 1.3). Even when the requisite number was as high as twelve or sixteen, those clever rat accountants continued to keep their registers up to date!

Two details are worth mentioning. First, the rats often squeezed lever A a little more than the minimum required—five times instead of four, for instance. Again, this was an eminently rational strategy. Since they received a penalty for switching prematurely to lever B, the rats preferred to play it safe and press lever A once more, rather than once less. Second, even after considerable training, the rats' behavior remained rather imprecise. Where the optimal strategy would have been to press lever A exactly four times, the rats often pressed it four, five, or six times, and on some trials they squeezed it three or even seven times. Their behavior was definitely not "digital," and variation was considerable from trial to trial. Indeed, this variability increased in direct proportion to the target number that the rats estimated. When the target number of presses was four, the rats' responses ranged from three to seven presses, but when the target was sixteen, the responses went from twelve to twenty-four, thus covering a much larger interval. The rats

[3] Mechner, 1958; Platt & Johnson, 1971

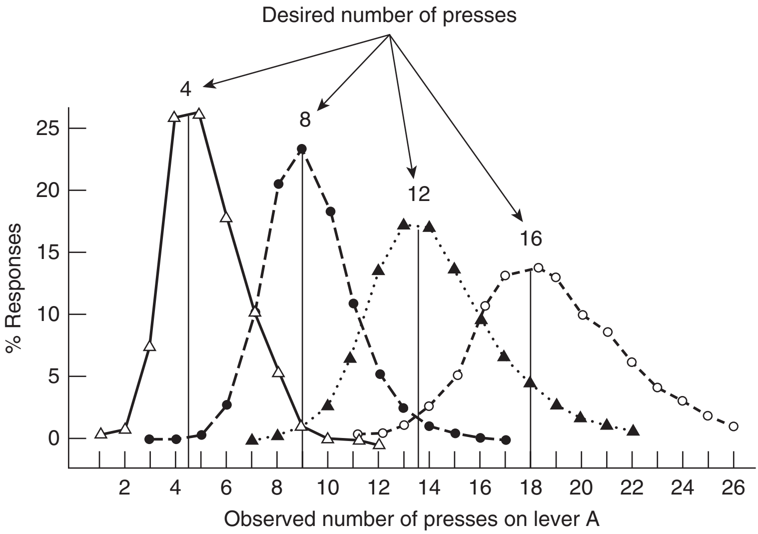

FIGURE 1.3. In an experiment by Mechner, a rat learns to press lever A a predetermined number of times before turning to a second lever B. The rat matches approximately the number selected by the experimenter, although its estimate becomes increasingly variable as the numbers get larger. (Adapted from Mechner 1958 by permission of the author and publisher; copyright © 1958 by the Society for the Experimental Analysis of Behavior.)

appeared to be equipped with a rather imprecise estimation mechanism, quite different from our digital calculators.

At this stage, many of you are probably wondering whether I am not too liberal in attributing numerical competence to rats, and whether a simpler explanation of their behavior might not be found. Let me first remark that the Clever Hans effect cannot have any influence on this type of experiment, because the rats are isolated in their cages and because all experimental events are controlled by an automated mechanical apparatus. However, is the rat really sensitive to the *number* of times the lever is pressed, or does it estimate the *time* elapsed since the beginning of a trial, or some other nonnumerical parameter? If the rat pressed at a regular rate, for instance once per second, then the above behavior might be fully explained by temporal rather than numerical estimation. While pressing on lever A, the rat would wait four, eight, twelve, or sixteen seconds, depending on the imposed schedule, before switching to lever B. This explanation might be considered simpler than the hypothesis that rats can count their movements—although, in fact, estimating duration and numbers are equally complex operations.

To refute such a temporal explanation, Francis Mechner and Laurence Guevrekian[4] used a very simple control: They varied the degree of food deprivation imposed on the rats. When the rats are really hungry, and therefore eager to obtain their food reward as

[4] Mechner & Guevrekian, 1962

fast as possible, they press the levers much faster. Nevertheless, this increase in rate has absolutely no effect on the *number* of times they press the lever. The rats that are trained with a target number of four presses continue to produce between three and seven presses, while the rats trained to squeeze eight times continue to squeeze about eight times, and so on. Neither the average number of presses, nor the dispersion of the results, is modified with higher rates. Obviously, a numerical rather than a temporal parameter drives the rats' behavior.

A more recent experiment by Russell Church and Warren Meck, at Brown University, demonstrates that rats spontaneously pay as much attention to the number of events as to their duration. In Church and Meck's experiment,[5] a loudspeaker placed in the rats' cage presented a sequence of tones. There were two possible sequences. Sequence A was made up of two tones and lasted a total of two seconds, whereas sequence B was made up of eight tones and lasted eight seconds. The rats had to discriminate between the two melodies. After each tune, two levers were inserted in the cage. To receive a food reward, the rats had to press the left lever if they had heard sequence A, and the right if they had heard sequence B (see Figure 1.4).

Several preliminary experiments had shown that rats placed in this situation rapidly learned to press the correct lever. Obviously, they could use two distinct parameters to distinguish A from B: the total duration of the sequence (two versus eight seconds) or the number of tones (two versus eight). Did rats pay attention to duration, number, or both? In order to find out, the experimenters presented some test sequences in which duration was fixed while number was varied, and others in which number was fixed while duration was varied. In the first case, all sequences lasted four seconds, but were made up of from two to eight tones. In the second case, all sequences were made up of four tones, but duration extended from two to eight seconds. On all such test sequences, the rats always received a food reward, regardless of the lever they picked. In anthropocentric terms, the researchers were simply asking what these new stimuli sounded like to the rats, without letting the reward interfere with their decision. The experiment therefore measured the rats' ability to generalize previously learned behaviors to a novel situation.

The results are clear-cut. Rats generalized just as easily on duration as on number. When duration was fixed, they continued to press the left lever when they heard two tones, and the right lever when they heard eight tones. Conversely, when number was fixed, they pressed left for two-second sequences, and right for eight-second sequences. But what about intermediate values? Rats apparently reduced them to the closest stimulus that they had learned. Thus, the new three-tone sequence elicited the same response as the two-tone sequence used for training, while sequences with five or six tones were classified just as the original sequence of eight tones had been. Curiously, when the sequence comprised just four tones, the rats could not decide whether they should press

[5] Church & Meck, 1984

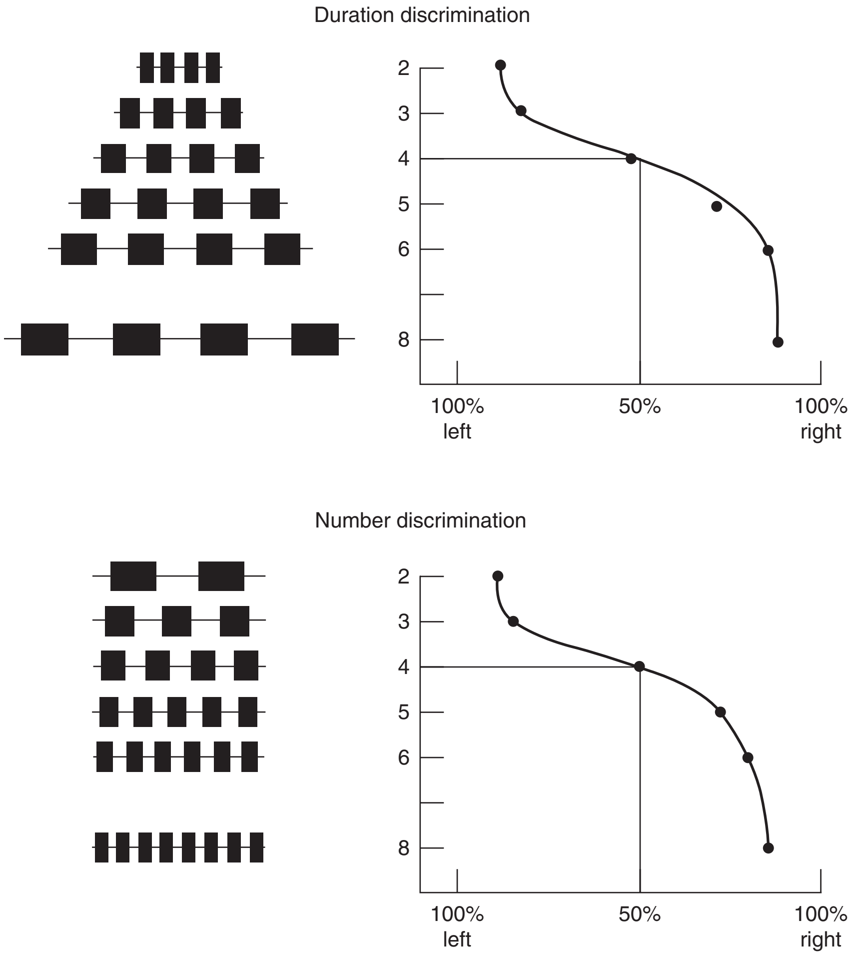

FIGURE 1.4. Meck and Church trained rats to press a lever on the left when they heard a short two-tone sequence, and a lever on the right when they heard a long eight-tone sequence. Subsequently, the rats generalized spontaneously: for equal numbers of sounds, they discriminated two-second sequences from eight-second sequences (top panel), and for an equal total duration, they discriminated two tones from eight tones (bottom panel). In both cases, four seems to be the "subjective middle" of 2 and 8, the point where rats cannot decide whether they should press right or left.
(Adapted from Meck and Church 1983.)

left or right. For a rat, four appears to be the subjective midpoint between the numbers two and eight!

Keep in mind that the rats did not know during training that they would be tested subsequently with sequences that varied in duration or in number of tones. Hence, this experiment shows that when a rat listens to a melody, its brain simultaneously and spontaneously registers both the duration and the number of tones. It would be a serious mistake to think that because these experiments use conditioning, they somehow teach the rats how to count. On the contrary, rats appear on the scene with state-of-the-art hardware for visual, auditory, tactile, and numerical perception. Conditioning merely teaches the animal to associate perceptions that it has always experienced, such as representations of stimulus duration, color, or number, with novel actions such as pressing a lever. There is no reason to think that number is a complex parameter of the external world, one that is more abstract than other so-called objective or physical parameters such as color, position in space, or temporal duration. In fact, provided that an animal is equipped with the appropriate cerebral modules, computing the approximate number of objects in a set is probably no more difficult than perceiving their colors or their positions.

Indeed, we now know that rats and many other species spontaneously pay attention to numerical quantities of all kinds—actions, sounds, light flashes, food morsels.[6] For instance, researchers have proved that raccoons, when presented with several transparent boxes with grapes inside, can learn to systematically select those that contain three grapes and to neglect those that contain two or four. Likewise, rats have been conditioned to systematically take the fourth tunnel on the left in a maze, regardless of the spacing between consecutive tunnels. Other researchers have taught birds to pick the fifth seed that they find when visiting several interconnected cages. And pigeons can, under some circumstances, estimate the number of times they have pecked at a target and can discriminate, for instance, between forty-five and fifty pecks. As a final example, several animals, including rats, appear to remember the number of rewards and punishments that they have received in a given situation. An elegant experiment by E. J. Capaldi and Daniel Miller at Purdue University has even shown that when rats receive food rewards of two different kinds—say, raisins and cereals—they keep in mind three pieces of information at the same time: the number of raisins they have eaten, the number of pieces of cereals, and the total number of food items.[7] In brief, far from being an exceptional ability, arithmetic is quite common in the animal world. The advantages that it confers for survival are obvious. The rat that remembers that its hideout is the fourth to the left will move faster in the dark maze of tunnels that it calls home. The squirrel that notices that a branch bears two nuts, and neglects it for another one that bears three, will have more chances of making it safely through the winter.

How Abstract Are Animal Calculations?

When a rat presses a lever twice, hears two sounds, and eats two seeds, does it recognize that these events are all instances of the number "2"? Or can't it see the link between

[6] For reviews of numerical cognition in animals, see Davis & Pérusse, 1988; Gallistel, 1989; Gallistel, 1990; Brannon & Terrace, 1998; Dehaene, Dehaene-Lambertz, & Cohen, 1998; Cantlon & Brannon, 2007; Jacob & Nieder, 2008; Nieder & Dehaene, 2009

[7] Capaldi & Miller, 1988

numbers that are perceived through different sensory modalities? The ability to generalize across different modalities of perception or action is an important component of what we call the *number concept.* Let us suppose, as an admittedly extreme case, that a child systematically utters the word "four" whenever he or she sees four objects, but randomly picks the words "three," "four," or "nine" when he or she hears four sounds or makes four jumps. Although performance is no doubt excellent with visual stimuli, we would be reluctant to grant the child knowledge of the concept of "4", because we consider possession of this concept to entail being able to apply it to many different multimodal situations. As a matter of fact, as soon as children have learned a number word, they can immediately use it to count their toy cars, the meows of their cat, or the misdemeanors of their little brother. What about rats? Is their numerical competence confined to certain sensory modalities, or is it abstract?

Unfortunately, any answer must remain tentative because few successful experiments have been done on multimodal generalization in animals. However, Russell Church and Warren Meck[8] have shown that rats represent number as an abstract parameter that is not tied to a specific sensory modality, be it auditory or visual. They again placed rats in a cage with two levers, but this time stimulated them with visual as well as with auditory sequences. Initially, the rats were conditioned to press the left lever when they heard two tones, and the right lever when they heard four tones. Separately, they were also taught to associate two light flashes with the left lever, and four light flashes with the right lever. The issue was, how were these two learning experiences coded in the rat brain? Were they stored as two unrelated pieces of knowledge? Or, had the rats learned an abstract rule such as "2 is left, and 4 is right"? To find out, the two researchers presented mixtures of sounds and light flashes on some trials. They were amazed to observe that when they presented a single tone synchronized with a flash, a total of two events, the rats immediately pressed the left lever. Conversely, when they presented a sequence of two tones synchronized with two light flashes, for a total of four events, the rats systematically pressed the right lever. The animals generalized their knowledge to an entirely novel situation. Their concepts of the numbers "2" and "4" were not linked to a low level of visual or auditory perception.

Consider how peculiar the rats' behavior was on trials with two tones synchronized with two light flashes. Remember that in the course of their training, the rats were always rewarded for pressing the left lever after hearing two tones, and likewise after seeing two flashes of light. Thus, both the auditory "two tones" stimulus and the visual "two flashes" stimulus were associated with pressing the left lever. Nevertheless, when these two stimuli were presented together, the rats pressed the lever that had been associated with the number 4! To better grasp the significance of this finding, compare it with a putative experiment in which rats are trained to press the left lever whenever they see a square

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[8] Church & Meck, 1984

(as opposed to a circle), and to respond left whenever they see the color red (as opposed to green). If the rats were presented with a red square—the combination of both stimuli—I bet that they would press even more resolutely on the left lever. Why are the numbers of tones and flashes grasped differently from shapes and colors? The experiment demonstrates that rats "know," to some extent, that numbers do not add up in the same way as shapes and colors. A square plus the color red makes a red square, but two tones plus two flashes do not evoke an even greater sensation of twoness. Rather, 2 plus 2 makes 4, and the rat brain seems to appreciate this fundamental law of arithmetic.

Perhaps the best example of abstract addition abilities in an animal comes from work done by Guy Woodruff and David Premack at the University of Pennsylvania.[9] They set out to prove that a chimpanzee could do arithmetic with simple fractions. In their first experiment, the chimpanzee's task was simple: It was rewarded for selecting, among two objects, the one that was physically identical to a third one. For instance, when presented with a glass half-filled with a blue liquid, the animal had to point toward the identical glass when presented next to another glass that was filled up to three-quarters of its volume. The chimp immediately mastered this simple physical matching task. Then the decision was progressively made more abstract. The chimp might be shown a half-full glass again, but now the options were either half an apple or three-quarters of an apple. Physically speaking, both alternatives differed widely from the sample stimulus; yet the chimpanzee consistently selected the half apple, apparently basing its responses on the conceptual similarity between half a glass and half an apple. Fractions of one-quarter, one-half, and three-quarters were tested with similar success: The animal knew that one-quarter of a pie is to a whole pie as one-quarter of a glass of milk is to a full glass of milk.

In their last experiment, Woodruff and Premack showed that chimpanzees could even mentally combine two such fractions: When the sample stimulus was made of one-quarter apple and one-half glass, and the choice was between one full disc or three-quarters disc, the animals chose the latter more often than chance alone would predict. They were obviously performing an internal computation not unlike the addition of two fractions: $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. Presumably, they did not use sophisticated symbolic calculation algorithms as we would. But they clearly had an intuitive grasp of how these proportions should combine.

A final anecdote concerning Woodruff and Premack's work: Though the manuscript reporting their work was initially titled "Primitive mathematical concepts in the chimpanzee: proportionality and numerosity," an editorial error made it appear in the pages of the scientific journal *Nature* under the heading *"Primative* mathematical concepts"! Involuntary as it was, this alteration was not so improper. For primitive, indeed, the animal's ability was not. And if "primative" was taken to mean "specific to primates," then the neologism seemed very appropriate here, because such an abstract ability to add fractions has not been observed in any other species so far.

[9] Woodruff & Premack, 1981

Addition, however, is not the only numerical operation in the animal repertoire. The ability to compare two numerical quantities is an even more fundamental ability, and indeed it is widespread among animals. Show a chimpanzee two trays on which you have placed several bits of chocolate.[10] On the first tray, two piles of chocolate chips are visible, one with four pieces, and the other with three pieces. The second tray contains a pile with five pieces of chocolate and, separate from it, a single piece. Leave the animal enough time to watch the situation carefully before letting it choose one tray and eat its content. Which tray do you think that it will pick? Most of the time, without training, the chimpanzee selects the tray with the largest total number of chocolate chips (see Figure 1.5). Hence, the greedy primate must spontaneously compute the total of the first tray ($4 + 3 = 7$), then the total of the second tray ($5 + 1 = 6$), and finally it must reckon that 7 is larger than 6 and that it is therefore advantageous to choose the first tray. If the chimp could not do the additions, but was content with choosing the tray with the largest single pile of chocolates, it should have been wrong in this particular example because, while the pile with five chips on the second tray exceeds each of the piles on the first tray, the total amount of chips on the first tray is larger. Clearly, the two additions and the final comparison operation are all required for success.

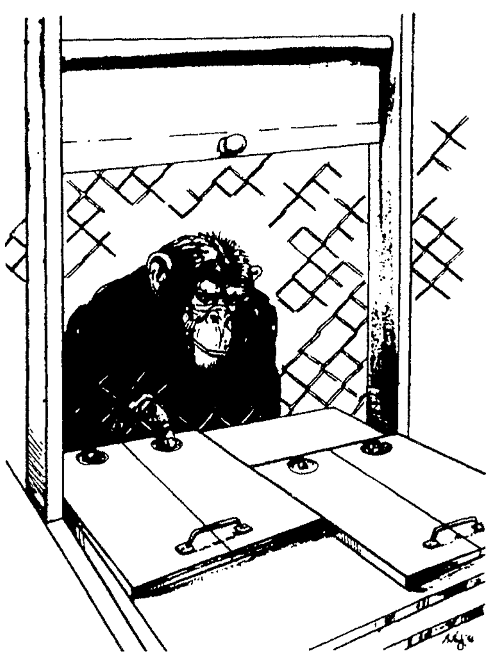

FIGURE 1.5. A chimpanzee spontaneously selects the pair of trays with the greater total number of chocolate bits, revealing its inborn ability to add and compare approximate numerosities. (Reprinted from Rumbaugh et al. 1987.)

[10] Rumbaugh, Savage-Rumbaugh, & Hegel, 1987

Although chimps perform remarkably well in selecting the larger of two numbers, their performance is not devoid of errors. As is frequently the case, the nature of these errors provides important cues about the nature of the mental representation employed.[11] When the two quantities are quite different, such as 2 and 6, chimpanzees hardly ever fail: They always select the larger. As the quantities become closer, however, performance systematically decreases. When the two quantities differ by only one unit, only 70% of the chimp's choices are correct. This systematic dependency of error rate on the numerical separation between the items is called the *distance effect*. It is also accompanied by a *magnitude effect*. For equal numerical distances, performance decreases as the numbers to be compared become larger. Chimpanzees have no difficulty in determining that 2 is larger than 1, even though these two quantities differ only by one unit. However, they fail increasingly more often as one moves to larger numbers such as 2 versus 3, 3 versus 4, and so on. Similar distance and magnitude effects have been observed in a great variety of tasks and in many species, including pigeons, rats, dolphins, and apes. No animals seem able to escape these laws of behavior—including, as we shall see later, *Homo sapiens*.

Why are these effects of distance and magnitude important? Because they demonstrate, once again, that animals do not possess a digital or discrete representation of numbers. Only the first few numbers—1, 2, and 3—can be discriminated with high accuracy. As soon as one advances toward larger quantities, fuzziness increases. The variability in the internal representation of numbers grows in direct proportion to the quantity represented. This is why, when numbers get large, an animal has problems distinguishing number n from its successor $n + 1$. One should not conclude, however, that large numbers are out of reach of the rat or pigeon brain. In fact, when numerical distance is sufficiently large, animals can successfully discriminate and compare very large numbers, on the order of 45 versus 50. Their imprecision simply leaves them blind to the finesses of arithmetic such as the difference between 49 and 50.

Within the limits set by this internal imprecision, we have seen through numerous examples that animals possess functional mathematical tools. They can add two quantities and spontaneously choose the larger of two sets. Should we really be that surprised? Let us first try to think whether the outcome of these experiments could possibly have been any different. When a hungry dog is offered a choice between a full dish and a half-full one of the same food, doesn't it spontaneously pick the larger meal? Acting otherwise would be devastatingly irrational. Choosing the larger of two amounts of food is probably one of the preconditions for the survival of any living organism. Evolution has been able to conceive such complex strategies for food gathering, storing, and predation, that it should not be astonishing that an operation as simple as the comparison of two quantities is available to so many species. It is even likely that a mental comparison algorithm was discovered early on, and perhaps even reinvented several times in the course of evolution.

[11] Dehaene, Dehaene-Lambertz et coll., 1998

Even the most elementary of organisms, after all, are confronted with a never-ending search for the best environment with the most food, the fewest predators, the most partners of the opposite sex, and so on. One must optimize in order to survive, and compare in order to optimize.

We still have to understand, however, by what neural mechanisms such calculations and comparisons are carried out. Are there minicalculators in the brains of birds, rats, and primates? How do they work?

The Accumulator Metaphor

How can a rat know that 2 plus 2 makes 4? How can a pigeon compare forty-five pecks with fifty? I know by experience that these results are often met with disbelief, laughter, or even exasperation—especially when the audience is composed of professors of mathematics! Our Western societies, ever since Euclid and Pythagoras, have placed mathematics at the pinnacle of human achievements. We view it as a supreme skill that either requires painful education, or comes as an innate gift. In many a philosopher's mind, the human ability for mathematics derives from our competence for language, so that it is inconceivable that an animal without language can count, much less calculate with numbers.

In this context, the observations about animal behavior that I have just described are in danger of being simply disregarded, as often happens with unexpected or seemingly aberrant scientific results. Without a theoretical framework to support them, they might appear as isolated findings—peculiar indeed, but eventually inconclusive and certainly not sufficient to question the equation "mathematics = language." To sort out such phenomena, we need a theory that explains, quite simply, how it is possible to count without words.

Fortunately, such a theory exists.[12] In fact, we all know of mechanical devices whose performances are not so different from those of rats. All cars, for instance, are equipped with a counting mechanism that keeps a record of the number of miles that have accumulated since the vehicle was first put in circulation. In its simplest version, this "counter" is just a cog wheel that advances by one notch for each additional mile. At least in principle, this example shows how a simple mechanical device may keep a record of an accumulated quantity. Why could a biological system not incorporate similar principles of counting?

The car counter is an imperfect example because it uses digital notation, a symbolic system that is most probably specific to humans. In order to account for the arithmetical abilities of animals, we should look for an even simpler metaphor. Imagine Robinson Crusoe, on his desert island, alone and helpless. For the sake of argument, let us even imagine that a blow to the head has deprived him of any language, leaving him unable to use number words for counting or calculation. How could Robinson build an approximate

[12] Meck & Church, 1983

calculator using only the makeshift means available to him? This is actually easier than it would seem. Suppose that Robinson has discovered a spring in the vicinity. He carves a tank from a large log, and places this accumulator next to the spring, so that water does not flow directly into it but can be temporarily diverted by using a small bamboo pipe. With this rudimentary device, of which the accumulator is the central component, Robinson will be able to count, add, and compare approximate numerical magnitudes. In essence, the accumulator enables him to master arithmetic as well as a rat or a pigeon does.

Suppose that a canoe loaded with cannibals approaches Robison's island. How can Robinson, who is following this scene with a telescope, keep a record of the number of attackers using his calculator? First, he would have to empty the accumulator. Then, each time a cannibal landed, Robinson would briefly divert some water from the spring into the accumulator. Furthermore, he does this so that it always takes a fixed amount of time and that the water flow remains constant throughout. Thus, for each attacker to be counted, a more or less fixed amount of water flows into the accumulator. In the end, the water level in the accumulator will be equal to n times the amount of water diverted at each step. This final water level may then serve as an approximate representation of the number n of cannibals who have landed. This is because it depends only on the number of events that have been counted. All other parameters, such as the duration of each event, the time interval between them, and so on, have no influence on it. The final level of water in the accumulator is thus completely equivalent to number.

By marking the level reached by water in the accumulator, Robinson can keep a record of how many people have landed, and he may use this number in later calculations. The next day, for instance, a second canoe approaches. To estimate the total number of attackers, Robinson first fills the accumulator up to the level of the preceding day's marker, and then adds a fixed amount of water for each newcomer, just as he did previously The new water level, after this operation is completed, will represent the result of the addition of attackers in the first canoe and in the second. Robinson can keep a permanent record of this computation by carving a different mark on the accumulator.

The day after, a few savages leave the island. To evaluate their number, Robinson empties his accumulator and repeats the above procedure, adding some water for each departing cannibal. He realizes that the final water level, which represents the number of people who have left, is much lower than the previous day's mark. By comparing the two water levels, Robinson reaches the worrisome conclusion that, in all likelihood, the number of natives that have left is smaller than the number of natives that have arrived in the past two days. In brief, Robinson, using his rudimentary device, can count, compute simple additions, and compare the results of his calculations, just like the animals in the above experiments.

A clear drawback of the accumulator is that numbers, although they form a discrete set, are represented by a continuous variable: water level. Given that all physical systems are inherently variable, the same number may be represented, at different times, by different amounts of water in the accumulator. Let us suppose, for instance, that water flow is not perfectly constant and varies randomly by between 4 and 6 liters per second, with a mean of 5 liters per second. If Robinson diverts water for two-tenths of a second into the accumulator, one liter on average will be transferred. However, this quantity will vary from 0.8 to 1.2 liters. Thus, if five items are counted, the final water level will vary by between 4 and 6 liters. Given that the very same levels could have been reached if four or six items had been counted, Robinson's calculator is unable to reliably discriminate the numbers 4, 5, and 6. If six cannibals land, and later only five depart, Robinson is in danger of failing to notice that one of them is missing. This, by the way, is exactly the situation that confronted the crow in the anecdote I mentioned at the beginning of this chapter! Robinson clearly will be better able to discriminate numbers that are more different; this is the distance effect. This effect will be exacerbated as the numbers become larger, thus reproducing the magnitude effect that also characterizes animal behavior.

One might object that the imaginary Robinson I am describing is not particularly clever. What prevents him from using marbles instead of imprecise amounts of water? Dropping in a bowl a single marble for each counted item would provide him with a discrete and precise representation of their number. In this manner, he would avoid errors even in the most complex of subtractions. But Robinson's machine is used here only as a metaphor for the animal brain. The nervous system—at least the one that rats and pigeons possess—does not seem to be able to count using discrete tokens. It is fundamentally imprecise, and seems unable to precisely keep track of the items that it counts; hence its increasing variance for larger and larger numbers.

Although the accumulator model is described here in a very informal manner, it is actually a rigorous mathematical model, the equations of which accurately predict variations in animal behavior as a function of number size and numerical distance.[13] The accumulator metaphor thus helps us to understand why rat behavior is so variable from one trial to the next. Even after considerable training, a rat seems unable to press exactly four times on a lever, but it can press four, five, or six times on different trials. I believe that this is due to a fundamental inability to represent numbers 4, 5, and 6 in a discrete and individualized format, as we do. To a rat, numbers are just approximate magnitudes, variable from time to time, and as fleeting and elusive as the duration of sounds or the saturation of colors. Even when an identical sequence of sounds is played twice, rats probably do not perceive the exact same number of sounds, but only the fluctuating level of an internal accumulator.

Of course, the accumulator is nothing more than a vivid metaphor that merely illustrates how a simple physical device can mimic, in considerable detail, experiments on animal arithmetic. There are no taps and recipients in the brains of rats and pigeons. Would it be possible, however, to identify, within the cerebrum, neuronal systems that might occupy a function similar to the components in the accumulator model? This is a completely open question. Currently, scientists are merely beginning to understand how

[13] Meck & Church, 1983, and for a more recent treatment, Dehaene, 2007

certain parameters are modified by various pharmacological substances. Injecting rats with metamphetamine, for instance, seems to accelerate the internal counter.[14] The rats injected with this substance respond to a sequence of four sounds as if they had been five or six. It is as if the flow of water to the accumulator were accelerated by metamphetamine. For each item counted, an amount of water larger than usual reaches the accumulator, thus making the final water level too great. This is how a 4 in the input may end up looking like a 6 at the output. We still have little knowledge, however, of the brain regions in which metamphetamine produces its accelerating effect. Cerebral circuitry is far from having revealed all its secrets.

Number-Detecting Neurons?

Although the cerebral circuits for number processing remain largely unknown, neural network simulations can be used to speculate on what their organization may be like. Neural network models are algorithms that run on a conventional digital computer, but emulate the kinds of computations that may go on in real brain circuits. Of course, the simulations are always vastly simplified when compared to the overarching complexity of real networks of neurons. In most computer models, each neuron is reduced to a digital unit with an output level of activation varying between 0 and 1. Active units excite or inhibit their neighbors, as well as more distant units, via connections with a variable weight, which are analogous to the synapses that connect real neurons. At each step, each simulated unit sums up the inputs it receives from other units, and switches on or off depending on whether the sum exceeds a given threshold. The analogy to a real nerve cell is crude, but one crucial property is preserved: the fact that a great many simple computations take place at the same time in several neurons distributed within multiple circuits. Most neurobiologists believe that such massive parallel processing is the key property that enables brains to perform complex computations in a short time using relatively slow and unreliable biological hardware.

Can parallel neuronal processing be used to process numbers? With Jean-Pierre Changeux, a neurobiologist at the Pasteur Institute in Paris, I have proposed a tentative neural network simulation of how animals extract numbers from their environment quickly and in parallel.[15] Our model addresses a simple problem that rats and pigeons routinely solve: given an input retina on which objects of various sizes are displayed, and given a cochlea on which tones of various frequencies are played, can a network of simulated neurons compute the total number of visual and auditory objects? According to the accumulator model, this number can be computed by adding to an internal accumulator

[14] For recent review, see Williamson, Cheng, Etchegaray, & Meck, 2008

[15] Dehaene & Changeux, 1993. This model has been later elaborated by others: Verguts & Fias, 2004; Verguts, Fias, & Stevens, 2005. See also Dehaene, 2007, and Pearson, Roitman, Brannon, Platt, & Raghavachari, 2010

a fixed quantity for each input item. The challenge is to do this with networks of simulated nerve cells, and to achieve a representation of number that is independent of the size and location of visual objects, as well as of the time of presentation of auditory tones.

We solved the problem by first designing a circuit that normalizes the visual input with respect to size. This network detects the locations occupied by objects on the retina, and allocates to each object, regardless of size and shape, an approximately constant number of active neurons on a location map. This normalization step is crucial because it allows the network to count each object as "one," regardless of size. As we shall see below, in mammals this operation may be achieved by circuits of the posterior parietal cortex, which are known to compute a representation of object location without taking exact shape and size into account.

In our simulation, a similar operation is also performed for auditory stimuli. Regardless of the time intervals at which they are received, auditory inputs are accumulated in a single memory store. Once these normalizations for size, shape, and time of presentation have been accomplished, it is easy to estimate number—one simply has to evaluate the total neuronal activity in the normalized visual map and in the auditory memory store. This total is equivalent to the final water level in the accumulator, and it provides a reasonably reliable estimate of number. In our simulation, the summation operation is taken care of by an array of units that pool activations from all the underlying visual and auditory units. Under certain conditions, these output units fire only when the total activity they receive falls within a predefined interval that varies from one neuron to the next. Each of these simulated neurons, therefore, works as a number detector that reacts only when a certain approximate number of objects is seen (Figure 1.6). One unit in the network, for instance, responds optimally when presented with four objects—be they, for instance, four visual blobs, four sounds, or two blobs and two sounds. The same unit reacts infrequently when presented with three or five objects, and not at all in the remaining cases. It therefore works as an abstract detector of number 4. The entire number line can be covered by such detectors, each tuned to a different approximate number, with the precision of tuning decreasing as one moves to increasingly larger numbers. Because the simulated neurons process all visual and auditory inputs simultaneously, the array of number detectors responds very quickly—it can estimate the cardinal of a set of four objects in parallel over the entire retina, without having to orient in turn toward each item as we do when we count.

Astonishingly, the number-detecting neurons that the model predicts seem to have been identified at least once in an animal brain. In the 1960s, Richard Thompson, a neuroscientist at the University of California at Irvine, recorded the activity of single neurons in the cortex of cats while the animals were presented with series of tones or of light flashes.[16] Some cells fired only after a certain number of events. One neuron, for instance, reacted after six events of any kind, regardless of whether this was six flashes of light,

[16] Thompson, Mayers, Robertson, & Patterson, 1970

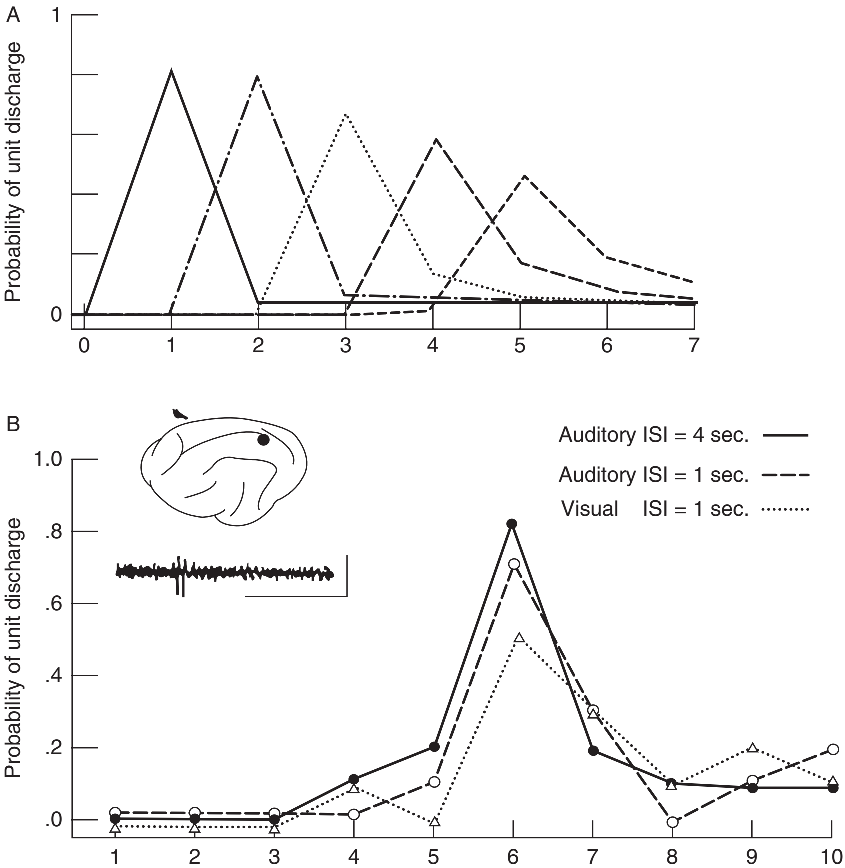

FIGURE 1.6. A computer-simulated neural network incorporates "numerosity detectors" that respond preferentially to a specific number of input items (top panel). Each curve shows the response of a given unit to different numbers of items. Note the decreasing selectivity of responses as input numerosity increases. In 1970, Thompson and his colleagues recorded similar "number-coding" neurons in the association cortex of anesthetized cats (bottom panel). The neuron illustrated here responds preferentially to six consecutive events, either six flashes of light one second apart, or six tones one or four seconds apart.

(Top, adapted from Dehaene and Changeux 1993; bottom, Thompson et al. 1970. Copyright © 1970 by American Association for the Advancement of Science).

six brief tones, or six longer tones. Sensory modality did not seem to matter: The neuron apparently cared only about number. Unlike a digital computer, it did not respond in a discrete all-or-none manner, either. Rather, its activation level grew after the fifth item, reached a peak for the sixth, and decreased for larger numbers of items, a response profile quite similar to that of the simulated neurons in our model. Several similar cells, each tuned to a different number, were recorded in a small area of the cat's cortex.

Thus, there might well be a specialized brain area, equivalent to Robinson's accumulator, in the animal brain. Unfortunately, Thompson's study, published in the prestigious